

Bound entanglement and distillability of multipartite quantum systems

Hui Zhao[†] Xin-yu Yu[†] Naihuan Jing[‡]

[†] *College of Applied Science, Beijing University of Technology, Beijing 100124, China and*

[‡] *Department of Mathematics, North Carolina State University, Raleigh, NC 27695, USA*

[‡] *School of Mathematical Sciences, South China*

University of Technology, Guangzhou 510640, China

(Dated: September 25, 2015)

Abstract

We construct a class of entangled states in $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ quantum systems with $\dim \mathcal{H}_A = \dim \mathcal{H}_B = \dim \mathcal{H}_C = 2$ and classify those states with respect to their distillability properties. The states are bound entanglement for the bipartite split $(AB) - C$. The states are NPT entanglement and 1-copy undistillable for the bipartite splits $A - (BC)$ and $B - (AC)$. Moreover, we generalize the results of $2 \otimes 2 \otimes 2$ systems to the case of $2n \otimes 2n \otimes 2n$ systems.

PACS numbers: 03.65.Ud, 03.67.Mn

Keywords: Bound entanglement; Distillability

I. INTRODUCTION

Quantum entanglement is one of the most astonishing quantum phenomena. It plays an important role in quantum information such as dense coding [1], quantum teleportation [2] and quantum cryptographic schemes [3–5].

Namely we say that a state of composite systems is considered to be entangled if it can not be written as a convex combination of product states [6]. Considerable efforts have been devoted to analyze the separability and entanglement [7–12]. Indeed there are two kinds of entangled states. One is the free entangled state which is distillable, and the other is the bound entangled state. A bound entangled state is one which is entangled and does not violate Peres condition [13]. For $2 \otimes 4$ and $3 \otimes 3$ systems, explicit examples of bound entangled states have been constructed in Ref. [14]. It has been shown that any state with PPT—positive partial transpose is non-distillable and a bipartite state is distillable if some number of copies $\rho^{\otimes n}$ can be projected to obtain a two-qubit state with NPT (non-PPT) [15]. Therefore the bound entanglement can not be brought to the singlet form by means of local quantum operations and classical communication from many copies of a given state. Instead, is an NPT state distillable? It was proved that for two-qubit systems all entangled states are distillable [16]. That means there is no NPT bound entangled state in $2 \otimes 2$ systems. For higher dimensions, the existence of bound entangled state with NPT has been discussed in Refs. [17–22]. As a matter of fact, bound entanglement can not be used alone for quantum communication, nevertheless, it can be distillable through free entanglement [23]. Moreover, in Ref. [24] for some bound entangled state ρ_1 with NPT there exists another bound entangled state ρ_2 such that the joint state $\rho_1 \otimes \rho_2$ is no longer a bound entangled state. Such a phenomenon is called superactivation.

In this paper, we analyze a class of tripartite entangled states. The paper is organized as follows. In Section 2, first we construct certain entangled states, then we give a detailed description about the entanglement with respect to different bipartite splits in $2 \otimes 2 \otimes 2$ systems. In Section 3, we generalize these results to $2n \otimes 2n \otimes 2n$ systems. Finally, conclusion and discussion are given in Section 4.

II. ENTANGLEMENT OF $2 \otimes 2 \otimes 2$ QUANTUM SYSTEMS

In this section we consider the entanglement of mixed states for different bipartite splits in $2 \otimes 2 \otimes 2$ systems. Consider the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, $\dim \mathcal{H}_A = \dim \mathcal{H}_B = \dim \mathcal{H}_C = 2$. Let $P_\phi = |\phi\rangle\langle\phi|$, e_i stand for orthonormal basis of \mathcal{C}^2 , $i = 1, 2$. We define the vectors

$$\begin{aligned}\Psi_1 &= \frac{1}{\sqrt{2}}(e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_1 \otimes e_2), \\ \Psi_2 &= \frac{1}{\sqrt{2}}(e_1 \otimes e_1 \otimes e_2 + e_2 \otimes e_2 \otimes e_1), \\ \Psi_3 &= \frac{1}{\sqrt{2}}(e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_2 \otimes e_2), \\ \Phi_b &= e_2 \otimes e_1 \otimes \left(\sqrt{\frac{1+b}{2}}e_1 + \sqrt{\frac{1-b}{2}}e_2 \right), \quad b \in [0, 1].\end{aligned}\tag{1}$$

We construct a state as following

$$\sigma_{insep} = \frac{2}{7} \sum_{i=1}^3 P_{\Psi_i} + \frac{1}{7} P_{e_1 \otimes e_2 \otimes e_2},\tag{2}$$

which is inseparable for all bipartite splits. It can be verified by using the partial transposition criterion. Now we define the following state

$$\sigma_b = \frac{7b}{7b+1} \sigma_{insep} + \frac{1}{7b+1} P_{\Phi_b}.\tag{3}$$

Its matrix is of the form

$$\sigma_b = \frac{1}{7b+1} \begin{pmatrix} b & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1+b}{2} & \frac{\sqrt{1-b^2}}{2} & 0 & 0 \\ b & 0 & 0 & 0 & \frac{\sqrt{1-b^2}}{2} & \frac{1-b}{2} & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & b \end{pmatrix}.\tag{4}$$

Next we analyze the inseparability of σ_b for all possible bipartite splits namely $(AB) - C$, $A - (BC)$, $B - (AC)$.

A. Bipartite split $(AB) - C$

For the bipartite split $(AB) - C$, we have

$$\sigma_b^{T_C} = \frac{1}{7b+1} \begin{pmatrix} b & 0 & 0 & 0 & 0 & 0 & 0 & b \\ 0 & b & 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & b & 0 \\ 0 & b & 0 & 0 & \frac{1+b}{2} & \frac{\sqrt{1-b^2}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{1-b^2}}{2} & \frac{1+b}{2} & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & b & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 0 & b \end{pmatrix}. \quad (5)$$

It is easy to see that the state $\sigma_b^{T_C}$ is positive as

$$\sigma_b^{T_C} = I \otimes I \otimes U \sigma_b I \otimes I \otimes U^\dagger, \quad (6)$$

where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (7)$$

We now prove that σ_b is an entangled state with respect to bipartite split $(AB) - C$ by using the range criterion. Assume that $b \neq 0, 1$, then any vector belonging to the range of σ_b can be presented as

$$u = (A_1, A_2, A_3, A_4, xA_5, A_1 + A_5, A_2, A_3), \quad A_i \in \mathcal{C}, i = 1, \dots, 5, \quad (8)$$

where $x = \sqrt{\frac{1+b}{1-b}}$. On the one hand, for $x \neq 0, 1$, if u is positive it must be of the form

$$u_{prod} = (r, s, t, q) \otimes (\tilde{A}_1, \tilde{A}_2) = (r\tilde{A}_1, r\tilde{A}_2, s\tilde{A}_1, s\tilde{A}_2, t\tilde{A}_1, t\tilde{A}_2, q\tilde{A}_1, q\tilde{A}_2), \quad (9)$$

where $r, s, t, q, \tilde{A}_1, \tilde{A}_2 \in \mathcal{C}$.

Comparing the two forms of vector u , we consider the following cases.

(i) If $rs \neq 0$, we can put $r = 1, s = 1$, then $A_1 = A_3 = \tilde{A}_1$, $A_2 = A_4 = \tilde{A}_2$, $q\tilde{A}_1 = A_2$, $q\tilde{A}_2 = A_3$, and $(q^2 - 1)\tilde{A}_2 = 0$. If $q^2 \neq 1$, then $\tilde{A}_1 = \tilde{A}_2 = 0$, $u = 0$. If $q^2 = 1$, we put $q = 1$, then $\tilde{A}_1 = \tilde{A}_2$, $xA_5 = t\tilde{A}_1$, $A_1 + A_5 = t\tilde{A}_1$, and $t = \frac{x}{x-1}$. We have

$$u_1 = A_1(1, 1, \frac{x}{x-1}, 1) \otimes (1, 1), \quad A_1 \in \mathcal{C}. \quad (10)$$

(ii) If $r \neq 0$, $s = 0$, we put $r = 1$, then $A_1 = \tilde{A}_1$, $A_2 = \tilde{A}_2$, $A_3 = A_4 = 0$, $q\tilde{A}_1 = A_2$, $q\tilde{A}_2 = 0$. For the case $q \neq 0$, we have $\tilde{A}_1 = \tilde{A}_2$, then $u = 0$. For $q = 0$, we get $A_2 = \tilde{A}_2 = 0$, $A_5 = -A_1$, we get

$$u_2 = A_1(1, 0, -x, 0) \otimes (1, 0), \quad A_1 \in \mathcal{C}. \quad (11)$$

(iii) If $r = 0$, $s \neq 0$, we put $s = 1$, then $A_1 = A_2 = q\tilde{A}_1 = 0$, $q\tilde{A}_2 = A_3 = \tilde{A}_1$. For $q \neq 0$, we have $\tilde{A}_1 = \tilde{A}_2 = 0$, then $u = 0$. For $q = 0$, we get $\tilde{A}_1 = 0$, $\tilde{A}_2 = A_4$, $A_5 = t\tilde{A}_2 = 0$, if $t \neq 0$, then $\tilde{A}_2 = 0$, $u = 0$. Then we have

$$u_3 = A_4(0, 1, 0, 0) \otimes (0, 1), \quad A_4 \in \mathcal{C}. \quad (12)$$

(iiii) If $r = 0$, $s = 0$, then $q\tilde{A}_1 = A_2 = 0$, $q\tilde{A}_2 = A_3 = 0$, $t\tilde{A}_1 = xA_5$, $t\tilde{A}_2 = A_5$. For $q \neq 0$, one has $\tilde{A}_1 = \tilde{A}_2 = 0$, $u = 0$. For $q = 0$,

$$u_{prod} = (0, 0, 0, 0, t\tilde{A}_1, t\tilde{A}_2, 0, 0), \quad (13)$$

if $t = 0$, then $u = 0$, we put $t = 1$, then

$$u_4 = A_5(0, 0, 1, 0) \otimes (x, 1), \quad A_5 \in \mathcal{C}. \quad (14)$$

All partial complex conjugations of vectors u_1, u_2, u_3, u_4 are

$$\begin{aligned} u_1^{\star 2} &= A_1(1, 1, \frac{x}{x-1}, 1) \otimes (1, 1), \\ u_2^{\star 2} &= A_1(1, 0, -x, 0) \otimes (1, 0), \\ u_3^{\star 2} &= A_4(0, 1, 0, 0) \otimes (0, 1), \\ u_4^{\star 2} &= A_5(0, 0, 1, 0) \otimes (x, 1). \end{aligned} \quad (15)$$

On the other hand, any vector belongs to the range of $\sigma_b^{T_C}$ can be written as

$$u' = (A'_1, A'_2, A'_3, A'_4, A'_2 + A'_5, xA'_5, A'_4, A'_1), \quad A'_i \in \mathcal{C}, i = 1, \dots, 5. \quad (16)$$

Let us check whether the vectors $u_1^{\star 2}, u_2^{\star 2}, u_3^{\star 2}, u_4^{\star 2}$ can be written in the above form. For $u_1^{\star 2}$, we obtain that $u_1^{\star 2}$ belongs to the rang of $(\sigma_b^{T_C})$. For $u_2^{\star 2}$, assuming that it is of the form u' , we get $A_1 = A'_1 = 0$, then $u_2^{\star 2}$ is the trivial zero vector. For $u_3^{\star 2}$, we have $A_4 = A'_4 = 0$, then $u_3^{\star 2}$ is the trivial zero vector. For $u_4^{\star 2}$, considering $A'_2 = 0$, $A'_2 + A'_5 = xA_5$, $xA'_5 = A_5$, we obtain $x^2 = 1$. This contradicts the fact that $x = \sqrt{\frac{1+b}{1-b}} \neq 0, 1$.

In summery, for any $b \neq 0, 1$, the state σ_b is a bound entangled state with respect to bipartite split $(AB) - C$.

B. Bipartite split $A - (BC)$

For the bipartite split $A - (BC)$, we have

$$\sigma_b^{T_{BC}} = \frac{1}{7b+1} \begin{pmatrix} b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & b & 0 \\ 0 & b & 0 & 0 & \frac{1+b}{2} & \frac{\sqrt{1-b^2}}{2} & 0 & 0 \\ 0 & 0 & b & 0 & \frac{\sqrt{1-b^2}}{2} & \frac{1+b}{2} & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b \end{pmatrix} \quad (17)$$

For any nonzero real vector $X = (x_1, x_2, \dots, x_8)^T$, we have

$$\begin{aligned} X^T \sigma_b^{T_{BC}} X &= f(x_1, x_2, \dots, x_8) = bx_1^2 + bx_8^2 - bx_5^2 + b(x_2 + x_5)^2 \\ &\quad + b(x_3 + x_6)^2 + b(x_4 + x_7)^2 + \left(\sqrt{\frac{1+b}{2}}x_5 + \sqrt{\frac{1-b}{2}}x_6\right)^2. \end{aligned} \quad (18)$$

Obviously, the positive index of inertia is 6, and the rank of $\sigma_b^{T_{BC}}$ is 7. Therefore σ_b is an NPT state with respect to bipartite split $A - (BC)$.

Next we will show that the state σ_b is 1-copy undistillable with respect to bipartite split $A - (BC)$. We begin with the following

Theorem 1. A bipartite state ρ acting on a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ is distillable if and only if there exist a positive integer $N \in \mathbb{N}$ and a Schmidt rank-2 state vector $|\psi_2^{[N]}\rangle$ in $\mathcal{H}_A^{\otimes N} \otimes \mathcal{H}_B^{\otimes N}$ such that [15]

$$\langle \psi_2^{[N]} | (\rho^{\otimes N})^{T_B} | \psi_2^{[N]} \rangle = \langle \psi_2^{[N]} | (\rho^{T_B})^{\otimes N} | \psi_2^{[N]} \rangle < 0. \quad (19)$$

For $N = 1$, the Schmidt rank-2 state is of the form

$$|\psi_2^{[1]}\rangle = \sum_{k,i=1}^2 \sum_{j=1}^4 c_k u_i^{(k)} v_j^{(k)} |i\rangle_A \otimes |j\rangle_{BC}, \quad (20)$$

where $\sum_{k=1}^2 c_k^2 = 1$, $\sum_{i=1}^2 u_i^{(k_1)*} u_i^{(k_2)} = \delta_{k_1 k_2}$, $\sum_{j=1}^4 v_j^{(k_1)*} v_j^{(k_2)} = \delta_{k_1 k_2}$. So we have

$$\langle \psi_2^{[1]} | \sigma_b^{T_{BC}} | \psi_2^{[1]} \rangle = \sum_{k_1, k_2, i=1}^2 \sum_{j=1}^4 \frac{1}{7b+1} c_{k_1}^* c_{k_2} u_i^{(k_1)*} (M_{(k_1, k_2)})_{i,j} u_i^{(k_2)} = \frac{1}{7b+1} Y_1^\dagger M_1 Y_1 \quad (21)$$

with $Y_1 = (c_1 u_1^1, c_1 u_2^1, c_2 u_1^2, c_2 u_2^2)^T$. We get the matrix M_1 is positive. According to the Theorem 1, the state σ_b is 1-copy undistillable with respect to bipartite split $A - (BC)$.

C. Bipartite split $B - (AC)$

For the bipartite split $B - (AC)$, we can use the same method as above, for any nonzero real vector $X = (x_1, x_2, \dots, x_8)^T$, we have

$$\begin{aligned} X^T \sigma_b^{T_{AC}} X = f(x_1, x_2, \dots, x_8) = & bx_2^2 + bx_7^2 - bx_5^2 + b(x_1 + x_6)^2 \\ & + b(x_3 + x_8)^2 + b(x_4 + x_5)^2 + \left(\sqrt{\frac{1+b}{2}}x_5 + \sqrt{\frac{1-b}{2}}x_6\right)^2. \end{aligned} \quad (22)$$

The positive index of inertia is 6, and the rank of $\sigma_b^{T_{AC}}$ is 7, then σ_b is also a NPT state with respect to bipartite split $B - (AC)$.

In the similar way, by direct calculation we have $\langle \psi_2^{[1]} | (\sigma_b^{T_{AC}} | \psi_2^{[1]}) \rangle \geq 0$ for all the Schmidt rank-2 states $|\psi_2^{[1]}\rangle$ in $\mathcal{H}_B^{\otimes 1} \otimes \mathcal{H}_{AC}^{\otimes 1}$. Therefore, σ_b is 1-copy undistillable with respect to bipartite split $B - (AC)$.

III. ENTANGLEMENT OF $2n \otimes 2n \otimes 2n$ QUANTUM SYSTEMS

Consider the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, $\dim \mathcal{H}_A = \dim \mathcal{H}_B = \dim \mathcal{H}_C = 2n$. Let $P_\phi = |\phi\rangle\langle\phi|$, e_i stand for orthonormal basis of \mathcal{C}^{2n} , $i = 1, 2, \dots, 2n$. We define the vectors

$$\begin{aligned} \Psi_{ijk} &= \frac{1}{\sqrt{2}}(e_i \otimes e_j \otimes e_k + e_{n+i} \otimes e_j \otimes e_{k+1}), \\ \Psi_{ik} &= \frac{1}{\sqrt{2}}(e_i \otimes e_k \otimes e_{2n} + e_{n+i} \otimes e_{k+1} \otimes e_1), \\ \Phi_a &= e_{n+1} \otimes e_1 \otimes \left(\sqrt{\frac{1+a}{2}}e_1 + \sqrt{\frac{1-a}{2}}e_{2n}\right), \quad a \in [0, 1]. \end{aligned} \quad (23)$$

where $i = 1, \dots, n$, $j = 1, \dots, 2n$, $k = 1, \dots, 2n-1$. Now we define the following state

$$\rho_{insep} = \frac{2}{8n^3 - 1} \sum_{i=1}^n \sum_{j=1}^{2n} \sum_{k=1}^{2n-1} (P_{\Psi_{ijk}} + P_{\Psi_{ik}}) + \frac{1}{8n^3 - 1} P_{\Phi_a}. \quad (24)$$

This state is inseparable with respect to all bipartite splits as there always exist a minor matrix of order 2 of its partial transposition is negative. Mixing the states ρ_{insep} and P_{Φ_a} , we have

$$\rho_a = \frac{(8n^3 - 1)a}{(8n^3 - 1)a + 1} \rho_{insep} + \frac{1}{(8n^3 - 1)a + 1} P_{\Phi_a}. \quad (25)$$

Next we analyze the different types of entanglement of ρ_a for all possible bipartite splits.

A. Bipartite split $(AB) - C$

For the bipartite split $(AB) - C$, $\rho_a^{T_C}$ is a $4n^2 \times 4n^2$ matrix

$$\rho_a^{T_C} = \frac{1}{(8n^3 - 1)a + 1} \begin{pmatrix} F_1 & 0 & \cdots & 0 & G_1^t & H_1^t & \cdots & 0 \\ 0 & F_1 & \cdots & 0 & 0 & G_1^t & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & F_1 & 0 & 0 & \cdots & G_1^t \\ G_1 & 0 & \cdots & 0 & K_1 & 0 & \cdots & 0 \\ H_1 & G_1 & \cdots & 0 & 0 & F_1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & G_1 & 0 & 0 & \cdots & F_1 \end{pmatrix}, \quad (26)$$

with

$$F_1 = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}, \quad G_1 = \begin{pmatrix} 0 & a & 0 & \cdots & 0 \\ 0 & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & a \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$H_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ a & 0 & \cdots & 0 \end{pmatrix}, \quad K_1 = \begin{pmatrix} \frac{1+a}{2} & 0 & \cdots & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & a & 0 \\ \frac{\sqrt{1-a^2}}{2} & 0 & \cdots & 0 & \frac{1+a}{2} \end{pmatrix},$$

where F_1, G_1, H_1, K_1 are all $2n \times 2n$ matrices and G^t stand for transposition of G .

For any nonzero real vector $X = (x_1, x_2, \cdots, x_{8n^3})^T$, we get

$$\begin{aligned} X^T \rho_a^{T_C} X &= \sum_{k=0}^{2n^2-1} \sum_{i=2}^{2n} a(x_{i+2nk} + x_{4n^3+i+2nk-1})^2 + \sum_{k=0}^{2n^2-2} a(x_{1+2nk} + x_{4n^3+4n+2nk})^2 \\ &\quad + ax_{4n^3-2n+1}^2 + \left(\sqrt{\frac{1-a}{2}} x_{4n^3+1} + \sqrt{\frac{1+a}{2}} x_{4n^3+2n} \right)^2. \end{aligned} \quad (27)$$

Obviously, the positive index of inertia is $4n^3 + 1$, and the rank of $\rho_a^{T_C}$ is $4n^3 + 1$. We drive that the state $\rho_a^{T_C}$ is a PPT state.

Next, we will show that the state ρ_a is entangled with respect to bipartite split $(AB) - C$. For any vector belongs to the range of ρ_a^{TC} can be presented as

$$v = (A_1, A_2, \dots, A_{2n-1}, A_{2n}, A_{2n+1}, \dots, A_{4n-1}, A_{4n}, \dots, A_{4n^3-2n+1}, \dots, A_{4n^3-1}, A_{4n^3}, \\ A_2 + B, A_3, \dots, A_{2n}, yB, A_{2n+2}, \dots, A_{4n}, A_1, \dots, A_{4n^3-2n+2}, \dots, A_{4n^3}, A_{4n^3-4n+1}), \quad (28)$$

where $y = \sqrt{\frac{1+a}{1-a}}$, $A_i, B \in \mathcal{C}, i = 1, 2, \dots, 4n^3$.

For $y \neq 0, 1$, if v is positive, it must be of the form

$$v_{prod} = (s_1, s_2, \dots, s_{4n^2}) \otimes (\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_{2n}), \quad s_i, \tilde{A}_j \in \mathcal{C}, i = 1, \dots, 4n^2, j = 1, \dots, 2n. \quad (29)$$

Let us now consider the following cases, comparing the two forms of vector v .

(i) While $s_1 = 0$, we have $s_m = s_{m+2n^2} = 0$ and $s_{4n^2} = 0, m = 2, 3, \dots, 2n^2 - 1$. The proof is in Appendix A. Hence if $s_{2n^2} = 0$, then $s_{2n^2+1} \neq 0$, otherwise $v = 0$, we can put $s_{2n^2+1} = 1$, then we get

$$v_1 = B(0, 0, \dots, 0, 1, 0, \dots, 0) \otimes (1, 0, \dots, 0, y). \quad (30)$$

If $s_{2n^2} \neq 0$, combine with $s_{4n^2}(\tilde{A}_1, \dots, \tilde{A}_{2n-1}) = s_{2n^2}(\tilde{A}_2, \dots, \tilde{A}_{2n})$ and $\frac{s_{2n^2+1}}{x}\tilde{A}_{2n} = s_{2n^2+1}\tilde{A}_1$ one has $s_{2n^2+1} = 0$, we put $s_{2n^2} = 1$, so we get

$$v_2 = A_{4n^3-2n+1}(0, 0, \dots, 1, 0, 0, \dots, 0) \otimes (1, 0, \dots, 0, 0). \quad (31)$$

(ii) While $s_1 \neq 0$, we put $s_1 = 1$, then $\tilde{A}_i = A_i, i = 1, 2, \dots, 2n$. According to the relation $A_k = s_{2n^2+1}A_{k-1}, 3 \leq k \leq 2n$, we have that if for some $k, A_k \neq 0, 2 \leq k \leq 2n$, then A_2, \dots, A_{2n} are not zero and $s_{2n^2+1} \neq 0$, if for some $k, A_k = 0, 2 \leq k \leq 2n$, then $A_2 = \dots = A_{2n} = 0, s_{2n^2+1} \neq 0$.

If $A_1 = 0$, from $A_1 = s_{2n^2+2}A_{2n}$, then $s_{2n^2+2} = 0, A_{2n} \neq 0$, otherwise $v = 0$, according to the conclusion of Appendix A and $s_{2n^2}A_{2n} = s_{4n^2}A_{2n-1}$, one has $s_{2n^2} = 0$, therefore

$$v_3 = A_2(1, 0, \dots, 0, s_{2n^2+1}, 0, \dots, 0) \otimes (0, 1, s_{2n^2+1}, s_{2n^2+1}^2, \dots, s_{2n^2+1}^{2n-2}). \quad (32)$$

If $A_1 \neq 0$, we put $s_{2n^2+1} = 1$, then $A_2 + B = A_1, yB = A_{2n}$ and $A_2 = \dots = A_{2n}$. From $A_1 = s_{2n^2+2}A_{2n}$, we obtain $s_{2n^2+2} = \frac{y+1}{y}$. Since $s_mA_2 = s_{2n^2+m}A_1, 2 \leq m \leq 2n^2$ and $s_mA_1 = s_{2n^2+m+1}A_{2n}, 2 \leq m \leq 2n^2 - 1$, then $s_m = (\frac{y+1}{y})^{2m-2}, s_{2n^2+m} = (\frac{y+1}{y})^{2m-3}$, we have

$$v_4 = A_2(1, (\frac{y+1}{y})^2, (\frac{y+1}{y})^4, \dots, (\frac{y+1}{y})^{4n^2-2}, 1, \frac{y+1}{y}, \dots, (\frac{y+1}{y})^{4n^2-3}) \\ \otimes (\frac{y+1}{y}, 1, 1, \dots, 1). \quad (33)$$

All partial complex conjugations of vectors v_1, v_2, v_3, v_4 are

$$\begin{aligned}
v_1^{\star 2} &= B(0, 0, \dots, 0, 1, 0, \dots, 0) \otimes (1, 0, \dots, 0, y), \\
v_2^{\star 2} &= A_{4n^3-2n+1}(0, 0, \dots, 1, 0, 0, \dots, 0) \otimes (1, 0, \dots, 0, 0), \\
v_3^{\star 2} &= A_2(1, 0, \dots, 0, s_{2n^2+1}, 0, \dots, 0) \otimes (0, 1, s_{2n^2+1}^*, s_{2n^2+1}^{2*}, \dots, s_{2n^2+1}^{2n-2*}), \quad s_{2n^2+1} \neq 0, \\
v_4^{\star 2} &= A_2(1, (\frac{y+1}{y})^2, (\frac{y+1}{y})^4, \dots, (\frac{y+1}{y})^{4n^2-2}, 1, \frac{y+1}{y}, \dots, (\frac{y+1}{y})^{4n^2-3}) \\
&\quad \otimes (\frac{y+1}{y}, 1, 1, \dots, 1).
\end{aligned} \tag{34}$$

On the other hand, any vector belongs to the range of ρ_a can be written as

$$\begin{aligned}
v' &= (A'_1, A'_2, \dots, A'_{2n}, \dots, A'_{4n^3-2n+1}, A'_{4n^3-2n+2}, \dots, A'_{4n^3}, yB', A'_1, \dots, A'_{2n-2}, \\
&\quad B' + A'_{2n-1}, A'_{2n}, A'_{2n+1}, \dots, A'_{4n-1}, \dots, A'_{4n^3-2n}, A'_{4n^3-2n+1}, \dots, A'_{4n^3-1}),
\end{aligned} \tag{35}$$

Now we check whether vectors $v_1^{\star 2}, v_2^{\star 2}, v_3^{\star 2}, v_4^{\star 2}$ can be written in the above form.

For $v_1^{\star 2}$, assume it can be written as the form of v' , we get $B = yB'$, $yB = B'$, then $y^2 = 1$, which contradicts the fact that $y \neq 0, 1$. For $v_2^{\star 2}$, we certainly have $A_{4n^3-2n+1} = A'_{4n^3-2n+1} = 0$, then $v_2^{\star 2}$ is the zero vector. For $v_3^{\star 2}$, it must be hold $s_{2n^2+1}A_2 = A'_1 = 0$, then $s_{2n^2+1} = 0$. This contradicts the fact that $s_{2n^2+1} \neq 0$. For $v_4^{\star 2}$, considering $A'_1 = \frac{y+1}{y}A_2$ and $A'_1 = A_2$, we get $A_2 = 0$, then $v_4^{\star 2}$ is also the zero vector.

Therefore, it leads to the conclusion that none of vectors $v_1^{\star 2}, v_2^{\star 2}, v_3^{\star 2}, v_4^{\star 2}$ belongs to the range of ρ_a . For any $a \neq 0, 1$, the state ρ_a is a bound entangled state with respect to bipartite split $(AB) - C$.

B. Bipartite split $A - (BC)$

For the bipartite split $A - (BC)$, we have $\rho_a^{T_{BC}}$ is a $2n \times 2n$ matrix

$$\rho_a^{T_{BC}} = \frac{1}{(8n^3 - 1)a + 1} \begin{pmatrix} F_2 & 0 & 0 & \dots & 0 & G'_2 & H'_2 & 0 & \dots & 0 \\ 0 & F_2 & 0 & \dots & 0 & 0 & G'_2 & H'_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & F_2 & 0 & 0 & 0 & \dots & G'_2 \\ G_2 & 0 & 0 & \dots & 0 & K_2 & 0 & 0 & \dots & 0 \\ H_2 & G_2 & 0 & \dots & 0 & 0 & F_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & G_2 & 0 & 0 & 0 & \dots & F_2 \end{pmatrix} \tag{36}$$

with

$$F_2 = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & a & 0 & \cdots & 0 \\ 0 & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & a \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ a & 0 & \cdots & 0 \end{pmatrix},$$

and

$$K_2 = \begin{pmatrix} \frac{1+a}{2} & 0 & \cdots & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & a & 0 & 0 & \cdots & 0 \\ \frac{\sqrt{1-a^2}}{2} & 0 & \cdots & 0 & \frac{1+a}{2} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & a & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a \end{pmatrix},$$

which are $4n^2 \times 4n^2$ matrices.

For any nonzero real vector $X = (x_1, x_2, \dots, x_{8n^3})^T$, we get

$$\begin{aligned} X^T \rho_a^{T_{BC}} X &= \sum_{k=0}^{n-1} \sum_{i=2}^{4n^2} a(x_{i+4kn^2} + x_{4n^3+4kn^2+i-1})^2 + \sum_{k=0}^{n-2} a(x_{1+4kn^2} + x_{4n^3+(8+4k)n^2})^2 \\ &+ (\sqrt{\frac{1-a}{2}} x_{4n^3+2n} + \sqrt{\frac{1+a}{2}} x_{4n^3+1})^2 + ax_{4n^3-4n^2+1}^2 + ax_{4n^3+4n^2}^2 - ax_{4n^3+1}^2. \end{aligned} \quad (37)$$

Obviously, the state $\rho_a^{T_{BC}}$ is not positive, so ρ_a is a NPT state with respect to the bipartite split $A - (BC)$.

Now we prove ρ_a is 1-copy undistillable with respect to the bipartite split $A - (BC)$ by using Theorem 1.

For $N = 1$, the Schmidt rank-2 state is of the form

$$|\varphi_2^{[1]}\rangle = \sum_{k=1}^2 \sum_{i=1}^{2n} \sum_{j=1}^{4n^2} c_k u_i^{(k)*} v_j^{(k)} |i\rangle_A \otimes |j\rangle_{BC}, \quad (38)$$

where $\sum_{k=1}^2 c_k^2 = 1$, $\sum_{i=1}^{2n} u_i^{(k_1)*} u_i^{(k_2)} = \delta_{k_1 k_2}$, $\sum_{j=1}^{4n^2} v_j^{(k_1)*} v_j^{(k_2)} = \delta_{k_1 k_2}$. Then we have

$$\begin{aligned} \langle \varphi_2^{[1]} | \rho_a^{T_{BC}} | \varphi_2^{[1]} \rangle &= \sum_{k_1, k_2=1}^2 \sum_{i=1}^{2n} \sum_{j=1}^{4n^2} \frac{1}{(8n^3 - 1)a + 1} c_{k_1}^* c_{k_2} u_i^{(k_1)*} (M_{(k_1, k_2)})_{i,j} u_i^{(k_2)} \\ &= \frac{1}{(8n^3 - 1)a + 1} Y_2^\dagger M_2 Y_2 \end{aligned} \quad (39)$$

with $Y_2 = (c_1 u_1^1, c_1 u_2^1, \dots, c_1 u_{2n}^1, c_2 u_1^2, c_2 u_2^2, \dots, c_2 u_{2n}^2)^T$, and the matrix M_2 are positive, that is $\langle \varphi_2^{[1]} | \rho_a^{T_{BC}} | \varphi_2^{[1]} \rangle \geq 0$ for any Schmidt rank-2 state vector $|\varphi_2^{[1]}\rangle$ in $\mathcal{H}_A^{\otimes 1} \otimes \mathcal{H}_{BC}^{\otimes 1}$. Therefore ρ_a is 1-copy undistillable with respect to the bipartite split $A - (BC)$.

C. Bipartite split $B - (AC)$

We can use the same method to analyze the case of bipartite split $B - (AC)$. For any nonzero real vector $X = (x_1, x_2, \dots, x_{8n^3})^T$, we get

$$\begin{aligned} X^T \rho_a^{T_{AC}} X &= \sum_{k=0}^{2n^2-1} \sum_{i=1}^{2n-1} a(x_{i+2kn} + x_{4n^3+2kn+i-1})^2 + \sum_{k=0}^{2n^2-2} a(x_{2n(2+k)} + x_{4n^3+2kn+1})^2 \\ &+ \left(\sqrt{\frac{1-a}{2}} x_{4n^3+2n} + \sqrt{\frac{1+a}{2}} x_{4n^3+1} \right)^2 + ax_{4n^3+2n+1}^2 + ax_{2n}^2 - ax_{4n^3+1}^2, \end{aligned} \quad (40)$$

then $\rho_a^{T_{AC}}$ is not positive, ρ_a is a NPT state with respect to the bipartite split $B - (AC)$. By direct calculation $\langle \varphi_2^{[1]} | \rho_a^{T_{AC}} | \varphi_2^{[1]} \rangle$ is positive, where $\varphi_2^{[1]} \in \mathcal{H}_B^{\otimes 1} \otimes \mathcal{H}_{AC}^{\otimes 1}$.

Therefore ρ_a is 1-copy undistillable with respect to bipartite split $B - (AC)$.

IV. CONCLUSION AND DISCUSSION

In summary, we have constructed a class of tripartite entangled states, then presented a detailed description about the entanglement with respect to all possible bipartite splits in $2 \otimes 2 \otimes 2$ systems. For the bipartite split $(AB) - C$, the state is bound entanglement, for another two bipartite splits, it is a NPT state and 1-copy undistillable. Finally, we have generalized the results to the case of $2n \otimes 2n \otimes 2n$ systems.

In order to avoid complicated calculations, we can also use the following method to prove 1-copy undistillation. According to the Ref. [25], a bipartite state ρ acting on a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ is distillable if and only if there exist a positive integer K and two 2-dimensional projectors $P : (\mathcal{H}_A)^{\otimes K} \rightarrow \mathbb{C}^2$ and $Q : (\mathcal{H}_B)^{\otimes K} \rightarrow \mathbb{C}^2$ such that $((P \otimes Q) \rho^{\otimes K} (P \otimes Q))^T$ is not positive. For example, in $2 \otimes 2 \otimes 2$ systems, let $\{|1\rangle, |2\rangle\}$ and $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$ be orthonormal bases of \mathcal{H}_A and \mathcal{H}_{BC} respectively, we take $K = 1$, considering the following two-dimensional projectors $P = |1\rangle\langle 1| + |2\rangle\langle 2|$ and $Q_1 = |1\rangle\langle 1| + |2\rangle\langle 2|$. Then the nonzero eigenvalues of matrix $((P \otimes Q_1) \sigma_b (P \otimes Q_1))^{T_{BC}}$ are $\frac{b}{7b+1}, \frac{1}{7b+1}(b \pm \frac{\sqrt{2b^2-2b+1}}{2} + \frac{1}{2})$, which are positive for $b \in (0, 1)$. For another possible two-dimensional projectors Q_i of \mathcal{H}_{BC} ,

$i = 2, 3, \dots, 6$, we also get the matrix $((P \otimes Q_i)\sigma_b(P \otimes Q_i))^{T_{BC}}$ is positive by calculating the eigenvalues, then σ_b is 1-copy undistillable with respect to the bipartite split $A - (BC)$. Using the same method, it is also easy to get σ_a is 1-copy undistillable with respect to the bipartite splits $B - (AC)$.

In $2n \otimes 2n \otimes 2n$ systems, let $K = 1$, according to the form of matrix $\rho_a^{T_{BC}}$, after taking every possible two-dimensional projectors P and Q of \mathcal{H}_A and \mathcal{H}_{BC} respectively, the nonzero rows and columns of matrix $(P \otimes Q)\rho_a^{T_{BC}}(P \otimes Q)$ constituting a new matrix J only has five kinds of form as following

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \begin{pmatrix} \frac{1+a}{2} & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \begin{pmatrix} a & 0 & 0 & a \\ 0 & a & a & 0 \\ 0 & a & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & a & 0 \\ 0 & a & \frac{1+a}{2} & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & \frac{1+a}{2} & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & \frac{\sqrt{1-a^2}}{2} & \frac{1+a}{2} \end{pmatrix}. \quad (41)$$

Obviously, the nonzero eigenvalues of matrix $(P \otimes Q)\rho_a^{T_{BC}}(P \otimes Q)$ are equal to the one of matrix J . It is easy to check that all eigenvalues of J are positive for $a \in (0, 1)$, then $(P \otimes Q)\rho_a^{T_{BC}}(P \otimes Q)$ is positive for all two-dimensional projectors P and Q . Therefore, ρ_a is 1-copy undistillable with respect to the bipartite split $A - (BC)$. Using the same method to analyze the case of bipartite split $B - (AC)$, we get ρ_a is also 1-copy undistillable.

We also hope that our results will help further investigations of multipartite quantum systems.

Appendix A

Comparing the two forms of v , we have

$$s_m(\tilde{A}_2, \tilde{A}_3, \dots, \tilde{A}_{2n}) = s_{2n^2+m}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_{2n-1}), \quad (A1)$$

$$s_{m-1}\tilde{A}_1 = s_{2n^2+m}\tilde{A}_{2n}, \quad (A2)$$

where $m = 2, 3, \dots, 2n^2$. In fact, we can obtain the two relations from (A1) and (A2),

- (i) if $s_{m-1} = 0$, then $s_{2n^2+m} = 0$, $m = 2, 3, \dots, 2n^2$,
- (ii) if $s_{2n^2+m} = 0$, then $s_m = 0$, $m = 2, 3, \dots, 2n^2 - 1$.

Let us prove the first one. Assume that $s_{m-1} = 0$ and $s_{2n^2+m} \neq 0$, $m = 2, 3, \dots, 2n^2$. From (A2), we have $\tilde{A}_{2n} = 0$, then $s_m\tilde{A}_1 = s_{2n^2+m+1}\tilde{A}_{2n} = 0$. Here, if $s_m \neq 0$, the $\tilde{A}_1 = 0$,

according to (A1), we get $\tilde{A}_2 = 0, \tilde{A}_3 = 0, \dots, \tilde{A}_{2n} = 0$, v is a zero vector, so $s_m = 0$. From (A1) and $s_{2n^2+m} \neq 0$, $\tilde{A}_1 = 0, \tilde{A}_2 = 0, \dots, \tilde{A}_{2n-1} = 0$ must hold, v is also a zero vector, so $s_{2n^2+m} = 0$.

For the second one, if $s_{2n^2+m} = 0$, and $s_m \neq 0$, $m = 2, 3, \dots, 2n^2 - 1$, then $\tilde{A}_2 = 0, \tilde{A}_3 = 0, \dots, \tilde{A}_{2n} = 0$. Since $s_m \tilde{A}_1 = s_{2n^2+m+1} \tilde{A}_{2n}$, we get $\tilde{A}_1 = 0$, then $v = 0$. Therefore if $s_{2n^2+m} = 0$, then $s_m = 0$.

ACKNOWLEDGMENTS

This work is supported by the China Scholarship Council, the National Natural Science Foundation of China (11271138, and 11275131), Beijing Natural Science Foundation Program and Scientific Research Key Program of Beijing Municipal Commission of Education (KZ201210028032) and the Importation and Development of High-Caliber Talents Project of Beijing Municipal Institutions (CITTCD201404067).

-
- [1] C.H. Bennett and S.J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
 - [2] C.H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres and W.K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
 - [3] A. Ekert, Phys. Rev. Lett. **67**, 661 (1991).
 - [4] D. Deutsch, A. Ekert, R. Jozsa, C. Macchiavello, S. Popescu and A. Sanpera, Phys. Rev. Lett. **77**, 2818 (1996).
 - [5] C.A. Fuchs, N. Gisin, R.B. Griffiths, C-S. Niu and A. Peres, Phys. Rev. A **56**, 1163 (1997).
 - [6] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).
 - [7] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
 - [8] H. Zhao, X.H. Zhang, S.M. Fei and Z.X. Wang, Chin. Sci. Bull. **58**, 2334 (2013).
 - [9] N. Brunner, J. Sharam and T. Vertesi, Phys. Rev. Lett. **108**, 110501 (2012).
 - [10] S.Q. Yan, Y. Guo and J.C. Hou, Chin. Sci. Bull. **59**, 279 (2014).
 - [11] W. Wen, Sci. China. Phys. Mech. Astron, **56**, 974 (2013).
 - [12] Y.Z. Wang, J.C. Hou and Y. Guo, Chin. Sci. Bull. **57**, 1643 (2012).
 - [13] A. Peres Phys. Rev. Lett. **77**, 1413 (1996).

- [14] P. Horodecki, Phys. Lett. A **232**, 333 (1997).
- [15] M. Horodecki, P. Horodecki and R. Horodecki, Phys. Rev. Lett. **80**, 5239 (1998).
- [16] M. Horodecki, P. Horodecki and R. Horodecki, Phys. Rev. Lett. **78**, 574 (1997).
- [17] M. Horodecki and P. Horodecki, Phys. Rev. A **59**, 4206 (1999).
- [18] D. P. DiVincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal and A. V. Thapliyal, Phys. Rev. A **61**, 062312 (2000).
- [19] W. Dr, J. I. Cirac, M. Lewenstein and D. Bruß, Phys. Rev. A **61**, 062313 (2000).
- [20] T. Eggeling, K. G. H. Vollbrecht, R. F. Werner and M. M. Wolf, Phys. Rev. Lett. **87**, 257902 (2001).
- [21] B. Kraus, M. Lewenstein and J. I. Cirac, Phys. Rev. A, **65**, 042327 (2002).
- [22] R. O. Vianna and A. C. Doherty, Phys. Rev. A **74**, 052306 (2006).
- [23] P. Horodecki, M. Horodecki and R. Horodecki, Phys. Rev. Lett. **82**, 1056 (1999).
- [24] P. W. Shor, J. A. Smolin and B. M. Terhal, Phys. Rev. Lett. **86**, 2681 (2001).
- [25] S. L. Braunstein, S. Ghosh and S. Severini, Ann. Comb. **10**, 3 (2006).